General method of controlling chaos

Zhao Hong, Yan Jie, Wang Jiao, and Wang Yinghai*
Department of Physics, Lanzhou University, Lanzhou 730000, People's Republic of China
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We present a mathematical framework for describing the allowable forms of perturbations of a control parameter for the purpose of controlling chaos. The present paper extends the method initially proposed by Ott, Grebogi, and Yorke [Phys. Rev. Lett. **64**, 1196 (1990)] and later extended by Romeiras *et al.* [Physica D **58**, 165 (1992)], allowing for a more general choice of feedback forms. Among the allowable feedback forms, those that do not include the coordinates of the desired control object explicitly provide a natural way to go about tracking, especially when the parameter changes are involuntary. Another benefit of the method is that the control can be implemented by the use of earlier states of the system as the feedback information. The generalized method can be conveniently used to deal with an experimental system in the absence of an *a priori* mathematical system model where the delay coordinates are used. These are illustrated by numerical examples in the paper.

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I. INTRODUCTION

Ott, Grebogi, and Yorke (OGY) [1] proposed that an unstable periodic orbit can be stabilized by making some small time-dependent perturbations on one of the adjustable parameters. This method was extended by Romeiras *et al.* [2]. For a discrete time system

$$\mathbf{z}_{i+1} = \mathbf{f}(\mathbf{z}_i, \mathbf{q}), \tag{1}$$

where $\mathbf{z}_i \in \mathbb{R}^n$ includes n coordinate components, $\mathbf{q} \in \mathbb{R}^m$ includes m parameter components, and $\mathbf{f}$ is sufficiently smooth in both $\mathbf{z}$ and $\mathbf{q}$. To stabilize a fixed point $\mathbf{z}_*$ of (1), the perturbations on an adjustable parameter p ($p \in \mathbf{q}$) around a value p_0 vary as

$$p_{i+1} = p_0 + \boldsymbol{\varepsilon}\cdot(\mathbf{z}_{i+1} - \mathbf{z}_*), \tag{2}$$

where p is a component of $\mathbf{q}$, $\boldsymbol{\varepsilon}$ is an n-dimensional vector to be determined. The solution to the problem of the determination of $\boldsymbol{\varepsilon}$, such that the eigenvalues of the fixed point have specified values, is well known from control systems theory and is called the "pole placement technique" (see, for example, Romeiras *et al.* [2]). Obviously, two basic elements are needed to implement the control, i.e., a previously obtained fixed point $\mathbf{z}_*$ about which the control is achieved and the coordinates of the current state used as prompt feedback information. Thus two questions resulting from the algorithm can be asked. (a) Since the fixed point is generally a function of the parameters, i.e., $\mathbf{z}_* = \mathbf{z}_*(\mathbf{q})$, how can one control the system whose one (or more) parameter(s) change(s) with time? (b) In the case that prompt feedback is inaccessible (e.g., in the case where the time spent on the feedback circuit cannot be neglected), how can one implement the control by using the earlier states as feedback information?

The goal of controlling chaos when the system parameters change with time is to track the desired unstable periodic orbits. The changes of system parameters can be divided into two situations, i.e., the changes can be controlled externally as wanted and the changes are involuntary or cannot be controlled externally (systems in such a case are called time-dependent systems in this paper). In the first situation, Gills *et al.* and Carroll *et al.* [3] reported a way of tracking a fixed point in the parameter space. Assuming the location of the fixed point is obtained at an initial parameter value, their tracking procedure can be expressed as follows: estimate the location of the fixed point at a small changed parameter, substitute the estimated value into OGY's control formula, and iterate the perturbed map several times (say, 100), then measure the mean of the fluctuation of the perturbed parameter p around p_0. Repeating the two steps again and again, one can find a value which minimizes the mean of the fluctuation and treat this value as the true value of the fixed point at the new parameter. Obviously, their tracking procedure is not applicable to the second situation because the parameters cannot stop at certain values to wait for the fixed point to be predicted. However, time-dependent systems are very general in practice, so finding a way to control this kind of system is an important problem.

Question (b) is also of practical importance since in some cases prompt feedback is not accessible (e.g., in the case that the time spent on the feedback circuit cannot be neglected). We are not aware of work making an effort in this direction.

Let the perturbations take the general form

$$p_{i+1} = g(\mathbf{z}_i, p_i), \tag{3}$$

where $g(\mathbf{z}, p)$ is sufficiently smooth in the neighborhood of the desired orbit and is called the control function in the following. Since (3) generally is not a simple feedback of the prompt state as is (2), the "pole placement technique" is generally not applicable. This paper provides

*Author to whom correspondence should be addressed.

a mathematical framework to deal with general forms of perturbation and therefore to solve the problems mentioned above. The key idea is that the activated parameter should be considered as an independent variable and thus the stability of a periodic orbit of the perturbed system ought to be discussed in the coordinate-parameter space.

The plan of the paper is as follows. In Sec. II, we give a mathematical framework for describing the allowable control functions, which is put forward first in the case that the equations of the system are known and then extended to deal with experimental systems in the absence of an *a priori* mathematical system model.

In Sec. III, several numerical examples are provided in order to illustrate the effectiveness of the method. Some are to illustrate the application of tracking the desired fixed point in the case that the changes of the system parameters can be adjusted externally and in the case that the system is time dependent. Special attention is focused on time-dependent systems, especially on time-dependent experimental systems in the absence of an *a priori* mathematical system model, in this part of the section. The other examples show how to control chaos by using earlier states as feedback information.

Several control functions used in the paper for different purposes are listed in the following. Their applicability will be understood after the general discussion in Sec. II. To fulfill the purpose of tracking an unstable fixed point, the control function can be chosen as

$$p_{i+1} = p_0 + \boldsymbol{\varepsilon}\cdot(\mathbf{z}_{i+1} - \mathbf{z}_i) + \varepsilon_{n+1}(p_i - p_0) \tag{4}$$

in the case where the equations of the system are known and as

$$p_{i+1} = p_0 + \boldsymbol{\varepsilon}\cdot(\mathbf{z}_{i+1} - \mathbf{z}_i) + \varepsilon_{n+1}(p_i - p_0) + \varepsilon_{n+2}(p_{i-1} - p_0) + \cdots + \varepsilon_{n+r+1}(p_{i-r} - p_0) \tag{5}$$

in the absence of an *a priori* mathematical system model, where p_{i-r} is the previous rth value of p_i. The control functions above provide a natural way to go about tracking, especially when the parameter changes are involuntary because they do not include $\mathbf{z}_*$ explicitly.

In order to control chaos by making use of earlier states as feedback information, we chose

$$p_{i+1} = p_0 + \boldsymbol{\varepsilon}\cdot(\mathbf{z}_{i-l} - \mathbf{z}_*) + \varepsilon_{n+1}(p_{i-l} - p_0) \tag{6}$$

as the control function, where $\mathbf{z}_{i-l}$ and p_{i-l} are the previous $(l+1)$th state and parameter, respectively (notice that the present time is $i+1$).

Finally, in Sec. IV we present the main conclusion and some discussions.

II. DESCRIPTION OF THE METHOD

A. The case of a known mathematical system model

For the sake of simplicity we also consider the discrete time dynamical system (1) and we shall describe the method through the application to the stabilization of a fixed point $\mathbf{z}_*$ (i.e., period 1 orbit) of the map $\mathbf{f}$. The consideration of periodic orbits of period larger than 1 is straightforward and is briefly discussed in Sec. IV. Let us assume the perturbations are added to the system by perturbing a parameter p ($p \in \mathbf{q}$) around a value p_0. Generally, p and the coordinates will couple to each other, so p should also be taken as a variable. The exact analysis of the dynamical behavior of the perturbed system should be made in the coordinate-parameter space. The dynamics of the perturbed system generally is determined by an $(n+1)$-dimensional map in the coordinate-parameter space:

$$\begin{aligned} \mathbf{z}_{i+1} &= \mathbf{f}(\mathbf{z}_i, p_i) , \\ p_{i+1} &= g(\mathbf{z}_i, p_i). \end{aligned} \tag{7}$$

We limit the form of g to ensure $(\mathbf{z}_*, p_0)$ be the fixed point of this $(n+1)$-dimensional system. The linear approximation of the system in the neighborhood of $(\mathbf{z}_*, p_0)$ is

$$\begin{pmatrix} \delta\mathbf{z}_{i+1} \\ \delta p_{i+1} \end{pmatrix} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{z}} & \frac{\partial \mathbf{f}}{\partial p} \\ \frac{\partial g}{\partial \mathbf{z}} & \frac{\partial g}{\partial p} \end{bmatrix} \begin{pmatrix} \delta\mathbf{z}_i \\ \delta p_i \end{pmatrix} . \tag{8}$$

The stability of the fixed point $(\mathbf{z}_*, p_0)$ is determined by the $(n+1)\times(n+1)$ matrix

$$\mathbf{T} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{z}} & \frac{\partial \mathbf{f}}{\partial p} \\ \frac{\partial g}{\partial \mathbf{z}} & \frac{\partial g}{\partial p} \end{bmatrix} . \tag{9}$$

When one speaks of stabilizing an unstable fixed point $\mathbf{z}_*$ of the original system, one actually means to make the fixed point $(\mathbf{z}_*, p_0)$ of the perturbed system stable. The point $(\mathbf{z}_*, p_0)$ will be an asymptotically stable fixed point of the perturbed system if the following two conditions are satisfied: (a) $g(\mathbf{z}_*, p_0) = p_0$, which guarantees $(\mathbf{z}_*, p_0)$ be the fixed point of the $(n+1)$-dimensional perturbed system; (b) all the eigenvalues have modulus less than unity. The two conditions can be used as criteria by which one can examine if a given control function is allowable or not.

Now we present a general algorithm for obtaining the allowable control functions. The first step is to determine the appropriate elements of the last row of the matrix (9) according to $n+1$ desired eigenvalues since the perturbations only influence this row of the matrix. Assuming

$$\begin{aligned} &|\mathbf{T}| = (-1)^{n+1} c_0, \\ &\sum_{i=1}^{n+1} |\mathbf{T}_i| = (-1)^{n+2} c_1, \\ &\sum_{i=1}^{n} \sum_{j=i+1}^{n+1} |\mathbf{T}_{ij}| = (-1)^{n+3} c_2, \\ &\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n+1} |\mathbf{T}_{ijk}| = (-1)^{n+4} c_3, \\ &\vdots \\ &\sum_{i=1}^{n+1} t_{ii} = (-1)^{2n+1} c_n, \end{aligned} \tag{10}$$

where $|\mathbf{T}|$ is the determinant of the matrix $\mathbf{T}$; $|\mathbf{T}_i|$ is the determinant obtained by eliminating the ith row and the ith column of the matrix $\mathbf{T}$; similarly, $|\mathbf{T}_{ij}|$ is the determinant obtained by eliminating the ith and the jth rows and the ith and jth columns of the matrix $\mathbf{T}$, and so on. $\sum t_{ii}$ is just the trace of the matrix $\mathbf{T}$ with elements t_{ij}. In fact, c_j $(j = 0, 1, \ldots, n)$ are the coefficients of the eigenequation

$$\lambda^{n+1} + c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0. \tag{11}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}$ be $n+1$ given eigenvalues of (9); then the relationship of the coefficients and the eigenvalues results in

$$\begin{aligned} c_n &= (-1)^1 \sum_{i=1}^{n+1} \lambda_i, \\ c_{n-1} &= (-1)^2 \sum_{i=1}^{n+1} \sum_{j,j>i}^{n+1} \lambda_i\lambda_j, \\ c_{n-2} &= (-1)^3 \sum_{i=1}^{n+1} \sum_{j,j>i}^{n+1} \sum_{k,k>j}^{n+1} \lambda_i\lambda_j\lambda_k, \\ &\vdots \\ c_0 &= (-1)^{n+1}\lambda_1\lambda_2\cdots\lambda_{n+1}, \end{aligned} \tag{12}$$

respectively. Equation (10) includes $n+1$ linear equations which contain $n+1$ elements [i.e., the elements of the last row of (9)] to be determined; thus the solution generally exists.

The second step is to find allowable control functions according to the $n+1$ determined elements. Noticing that the $n+1$ elements in fact are components of $(\frac{\partial g}{\partial \mathbf{z}}, \frac{\partial g}{\partial p})$, the control function g generally should include at least $n+1$ coefficients to be determined. However, the choice of an allowable form of g may still be various. This feature provides us with wide freedom to choose different control functions for different purposes. The perturbation form (2) is one of the allowable control functions though it has only n coefficients to be determined. In fact, the control function is a simple feedback of the linear combination of the current coordinates. Hence the rank of $\mathbf{T}$ is reduced to n, which causes the first equation of (10) to be trivially 0, and thus one of the $n+1$ eigenvalues to be always 0. Then the n-dimensional vector $\boldsymbol{\varepsilon}$ can be calculated by the remaining n equations of (10).

B. In the absence of a mathematical system model

In experimental studies of chaotic dynamical systems, delay coordinates are often used to represent the system states [4,5]. This is sometimes useful because it only requires measurements of the time series of a single scalar state variable which we denote $\xi(t)$. To obtain a map, one can take a Poincaré surface of section and obtain the discrete time series ξ_i $(i = 1, 2, 3, \ldots)$ of the variable $\xi(t)$. Using time delay coordinates with time delay τ and embedding dimension n, an n-dimensional coordinate vector is formed as follows: $\mathbf{z}_i = (\xi_{Ni}, \xi_{Ni-\tau}, \xi_{Ni-2\tau}, \ldots, \xi_{Ni-(n-1)\tau})$. Here, N is an integer. If $N \geq (n-1)\tau$, then the set of the components of $\mathbf{z}_i$ and the set of the components of $\mathbf{z}_{i+1}$ have no intersection. For example, let $n = 3$ and $\tau = 1$. If we choose $N = 3$, then $\mathbf{z}_i = (\xi_{3i}, \xi_{3i-1}, \xi_{3i-2})$ and $\mathbf{z}_{i+1} = (\xi_{3i+3}, \xi_{3i+2}, \xi_{3i+1})$ have no common components. Dressler and Nitsche [6] proved that in this case $\mathbf{z}_{i+1}$ depends not only on p_i but also on p_{i-1}. Otherwise, if $N \leq (n-1)\tau$, then the set of the components of $\mathbf{z}_i$ and the set of the components of $\mathbf{z}_{i+1}$ will have some common components. For example, let $n = 3$, $\tau = 1$, and choose $N = 1$; then $\mathbf{z}_i = (\xi_i, \xi_{i-1}, \xi_{i-2})$ and $\mathbf{z}_{i+1} = (\xi_{i+1}, \xi_i, \xi_{i-1})$ have two common components ξ_i and ξ_{i-1}. It can be proved that in this case $\mathbf{z}_{i+1}$ depends on p_i, p_{i-1}, p_{i-2}, and p_{i-3}. In fact, when $N \leq (n-1)\tau$, $\mathbf{z}_{i+1}$ generally depends not only on p_i but also on some (say, r) earlier parameter values, i.e., the reconstructed system can be written as

$$\mathbf{z}_{i+1} = \mathbf{f}(\mathbf{z}_i, p_i, p_{i-1}, \ldots, p_{i-r}). \tag{13}$$

Noticing that $(\mathbf{z}_i, p_i, p_{i-1}, \ldots, p_{i-r})$ of time i is mapped to $(\mathbf{z}_{i+1}, p_{i+1}, p_i, \ldots, p_{i-r+1})$ of time $i+1$, the dynamics of the perturbed system should be discussed in an $(n+r+1)$-dimensional coordinate-parameter space. Let the perturbation at time $i+1$ have the general form

$$p_{i+1} = g(\mathbf{z}_i, p_i, p_{i-1}, p_{i-2}, \ldots, p_{i-r}). \tag{14}$$

Noticing that $p_i, p_{i-1}, \ldots$ and p_{i-r} remain fixed in the process from time i to time $i+1$, the perturbed map should be

$$\begin{aligned} \mathbf{z}_{i+1} &= \mathbf{f}(\mathbf{z}_i, p_i, p_{i-1}, p_{i-2}, \ldots, p_{i-r}), \\ p_{i-r+1} &= p_{i-r+1}, \\ &\vdots \\ p_{i-1} &= p_{i-1}, \\ p_i &= p_i, \\ p_{i+1} &= g(\mathbf{z}_i, p_i, p_{i-1}, p_{i-2}, \ldots, p_{i-r}). \end{aligned} \tag{15}$$

Let the fixed point of the original system be $\mathbf{z}_*$; then the fixed point of the perturbed map must be $(\mathbf{z}_*, p_0, p_0, \ldots, p_0)$. Since the $(n+r+1) \times (n+r+1)$ linearized Jacobian matrix of (15) must have at least $n+r+1$ adjustable components to satisfy $n+r+1$ desired eigenvalues, the control function should include at least $n+r+1$ control coefficients to be determined. The Jacobian matrix of the perturbed system is

$$\mathbf{T} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{z}} & \frac{\partial \mathbf{f}}{\partial p_i} & \frac{\partial \mathbf{f}}{\partial p_{i-1}} & \cdots & \frac{\partial \mathbf{f}}{\partial p_{i-r}} \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ \frac{\partial g}{\partial \mathbf{z}} & \frac{\partial g}{\partial p_i} & \frac{\partial g}{\partial p_{i-1}} & \cdots & \frac{\partial g}{\partial p_{i-r}} \end{bmatrix}. \tag{16}$$

The elements except those in the last row of this Jacobian matrix are experimentally accessible. Therefore the method is applicable to experimental systems.

III. APPLICATIONS

A. For systems where some parameters change with time

1. The case of a known mathematical model

To fulfill our purpose, the control function can be chosen as (4). Obviously, the control function is not a simple prompt feedback. An important difference from (2) is that (4) does not include the fixed point to be stabilized explicitly. For $n+1$ desired eigenvalues, the $n+1$ coefficients $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1})$ included in (3) can be determined by the method of Sec. II at a given $\mathbf{q}$. The coefficients can be considered as a point in the $(n+1)$-dimensional coefficient space. For the consideration of continuity, there must exist a region in which any point ensures that the eigenvalues of the fixed point of the perturbed system have modulus smaller than unity and we call this region the control region in the following. Any point in the control region is suitable to implement the control. Most of the numerical examples in this paper are done by using a point in the control region which lets all the eigenvalues of the matrix (9) be 0, and, for convenience, we call this point the "center" point of the control region in the following. One can calculate the "center" point by simply setting $c_0 = c_1 = \cdots = c_n = 0$ in Eq. (10). The control region is determined by the fixed point $\mathbf{z}_*$ and thus related to $\mathbf{q}$. A given $\mathbf{q}$ determines a control region; then two values $\mathbf{q}$ and $\mathbf{q}+\boldsymbol{\delta}$ determine two control regions. If the difference $\boldsymbol{\delta}$ is small enough, the two control regions will have an intersecting part. Any point in the intersecting part can be used to stabilize not only $\mathbf{z}_*(\mathbf{q})$ but also $\mathbf{z}_*(\mathbf{q}+\boldsymbol{\delta})$. The evolution of the perturbed dynamical system can terminate at either of the two fixed points, which only depend on what the current values of the parameters are. This is possible because the control function given by (4) does not include the desired fixed point explicitly. In other words, though the "center" point calculated at $\mathbf{q}$ is not the "center" point at $\mathbf{q}+\boldsymbol{\delta}$, it will be within the intersecting part of the control region of $\mathbf{q}$ and the control region of $\mathbf{q}+\boldsymbol{\delta}$. Thus it is still suitable for the stabilization of the desired fixed point at the changed parameter $\mathbf{q}+\boldsymbol{\delta}$. Actually, the "center" point determined at $\mathbf{q}_*$ can be used to stabilize the fixed point within a certain parameter region around $\mathbf{q}_*$ (which is called the stabilizable region of the center point of $\mathbf{q}_*$). Hence, once a desired fixed point is obtained at $\mathbf{q}_*$ by any method, one can calculate the "center" point and apply it to stabilize the fixed point over a region around $\mathbf{q}_*$. If the system is time independent, i.e., the parameter $\mathbf{q}$ does not depend on the time, the control function (4) certainly can make the perturbed system terminate at the fixed point $(\mathbf{z}_*, p_0)$ when the adjustable parameter p is perturbed around the value p_0. If the parameters of the system can be externally adjusted along a trajectory in the parameter space, one can track the fixed point along the trajectory by (4). The tracking procedure is described as follows. First calculate the "center" point $\boldsymbol{\varepsilon}(\mathbf{q}_1)$ at $\mathbf{q}_1$ if $\mathbf{z}_*(\mathbf{q}_1)$ is obtained by any method, then adjust $\mathbf{q}$ to the next value $\mathbf{q}_2$, and iterate the perturbed map several times by use of $\varepsilon(\mathbf{q}_1)$ to make the perturbed system terminate at $\mathbf{z}_*(\mathbf{q}_2)$. The procedure can be repeated until $\mathbf{q}$ is close to the edge of the stabilizable region of $\boldsymbol{\varepsilon}(\mathbf{q}_1)$. Then one can recalculate the "center" point according to the located fixed point $\mathbf{z}_*(\mathbf{q})$ and continue the tracking procedure. One need not know the edge of the stabilizable region of $\boldsymbol{\varepsilon}(\mathbf{q})$ though it can be calculated analytically. The time to recalculate the "center" point can be determined according to the following fact: the time spent for terminating at the fixed point will be heavily increased when $\mathbf{q}$ is close to the edge. Thus the recalculation of the "center" point can be done by computer according to the desired terminating rate.

Let us take the logistic map $x_{i+1} = c - a x_i^2$ as an example to illustrate the algorithm. Assuming a and c are two parameters, let us perturb a around a value a_0 as

$$a_{i+1} = a_0 + \varepsilon_1(x_{i+1} - x_i) + \varepsilon_2(a_i - a_0). \tag{17}$$

We get a two-dimensional perturbed map: $x_{i+1} = c - a_i x_i^2$ and $a_{i+1} = a_0 + \varepsilon_1(x_{i+1} - x_i) + \varepsilon_2(a_i - a_0)$. The original system has two fixed points defined by $x_+ = -\frac{1}{2a} + \frac{\sqrt{1+4ac}}{2a}$ and $x_- = -\frac{1}{2a} - \frac{\sqrt{1+4ac}}{2a}$. Assume x_+ is the fixed point to be tracked along the parameter c. Fixing $a = 1.6$, Fig. 1 shows the control region at $c = 1$ (solid-line-enclosed region) and the control region at $c = 1.2$ (broken-line-enclosed region) in the ε_1-ε_2 plane. The point C and the point C' are the "center" points

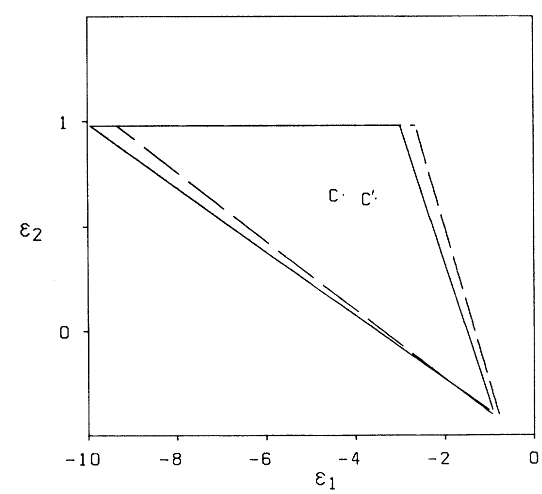

FIG. 1. The control regions around the "center" point at $c = 1.0$ (the solid-line-enclosed region) and at $c = 1.2$ (the broken-line-enclosed region), respectively. Point C and point C' are the corresponding "center" points.

for the parameter $c = 1$ and $c = 1.2$, respectively. One can see that these two regions intersect with each other, which means the fixed point can be stabilized by use of any point belonging to the intersecting part at least over the parameter range $c \in (1.0, 1.2)$. In fact, a direct calculation shows that the stabilizable range of the point C is $(-0.16, 2.05)$. Figure 2(a) shows that the fixed point is tracked just over the stabilizable range by use of the point C. In this tracking procedure, we add (subtract) 0.01 to (from) c at every change and then iterate the perturbed map five times to wait for the evolution of the system to terminate at the fixed point. In fact, when adding (subtracting) c on a scale larger (smaller) than 0.01 at every change, one gets the same result. This means that the range in which the fixed point can be tracked is just the range in which it can be stabilized. As a contrast, we also provide the bifurcation diagram of c in Fig. 2(a). One can see that the parameter range in which the fixed point can be tracked by a single "center" point is rather wide.

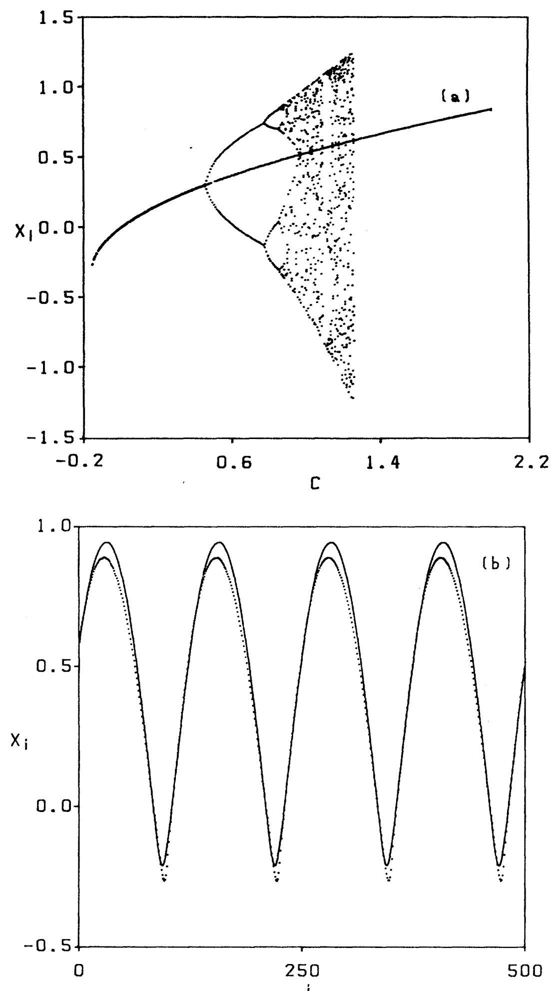

FIG. 2. (a) The bifurcation diagram of the logistic map $x_{i+1} = c - ax_i^2$ along the parameter c. The tracking result by using the "center" point C is also illustrated in the figure. (b) The result of tracking the fixed point x_+ when the underlying change of c is $c(i) = 1 + \delta \sin(\beta i)$ by using the "center" point C when $\delta = 1.15$ and $\beta = 0.05$. The trajectory of the perturbed system is denoted by the discrete points (the initial value is set at x_+ when $c = 1$ and $a = 1.6$). The solid line is $x_+(c)$.

More important, one can expect that the fixed points of time-dependent systems can also be tracked by use of the control function (4), as long as the change is slow enough. This is possible for the same reason that the perturbation form does not include the fixed point explicitly.

Let us still take the logistic map as an example, but this time we assume that c is time dependent intrinsically or cannot be adjusted externally. Assume the underlying change of c is $c(i) = 1 + \delta \sin(\beta i)$, where $c(i)$ denotes the value of c at time i, β determines the changing scale of c, and δ determines the changing range of c around 1. Fixing $a = 1.6$, Fig. 2(b) shows the results obtained by using the "center" point determined at $c = 1$ to track the fixed point of the system. In this figure, $\beta = 0.05$ and $\delta = 1.15$. One can see that the trajectory of the controlled system is very close to the desired object. Numerical results show that the point C is still effective even if $\beta = 0.1$ though the deviation from the desired object becomes more distinct.

2. In the absence of an a priori mathematical model

Now we illustrate how to implement the control for experimental systems in the absence of an *a priori* mathematical system model. The control function can be chosen as (5) for this purpose. Let us assume that the underlying system model is the Hénon map [7]

$$x_{i+1} = 1 - ax_i^2 + by_i, \quad y_{i+1} = x_i, \tag{18}$$

where x_i can be measured as a time series, the parameter a is the parameter to be perturbed, and b is the other parameter. This system has two fixed points defined by

$$x_+ = y_+ = \frac{b-1}{2a} + \frac{\sqrt{(b-1)^2 + 4a}}{2a}$$

and

$$x_- = y_- = \frac{b-1}{2a} - \frac{\sqrt{(b-1)^2 + 4a}}{2a} .$$

We assume that (x_+, y_+) is the desired control object.

In this example, if we choose $n = 2, \tau = 1$, and $N = 1$, then $\mathbf{z}_i = (x_i, x_{i-1})$ and $\mathbf{z}_{i+1} = (x_{i+1}, x_i)$ will have a common component x_i. As we mentioned in Sec. II, the situation makes $\mathbf{z}_{i+1}$ depend not only on a_i, a_{i-1} but also on a_{i-2}, and thus the perturbed system should be discussed in a five-dimensional coordinate-parameter space. The perturbations of a around the value a_0 should take the form

$$a_{i+1} = a_0 + \varepsilon_1(x_{i+1} - x_i) + \varepsilon_2(x_i - x_{i-1}) + \varepsilon_3(a_i - a_0) + \varepsilon_4(a_{i-1} - a_0) + \varepsilon_5(a_{i-2} - a_0). \tag{19}$$

If we choose $n = 2, \tau = 1$, and $N = 2$, then $\mathbf{z}_i = (x_{2i}, x_{2i-1})$ and $\mathbf{z}_{i+1} = (x_{2i+2}, x_{2i+1})$ will have no common component. So $\mathbf{z}_{i+1}$ depends only on a_i and a_{i-1} and the perturbed system is four dimensional. The corresponding control function is

$$a_{i+1} = a_0 + \varepsilon_1(x_{2i+2} - x_{2i}) + \varepsilon_2(x_{2i+1} - x_{2i-1}) + \varepsilon_3(a_i - a_0) + \varepsilon_4(a_{i-1} - a_0). \quad (20)$$

Similarly, if we choose $n = 2, \tau = 2$, and $N = 3$, then $\mathbf{z}_i = (x_{3i}, x_{3i-1})$ and $\mathbf{z}_{i+1} = (x_{3i+3}, x_{3i+2})$ have no common component and the control function is

$$a_{i+1} = a_0 + \varepsilon_1(x_{3i+3} - x_{3i}) + \varepsilon_2(x_{3i+2} - x_{3i-1}) + \varepsilon_3(a_i - a_0) + \varepsilon_4(a_{i-1} - a_0). \quad (21)$$

Effective control coefficients must exist for each of the three choices according to the discussion of Sec. II. However, the control ability of the first case is the best because the control is added to the system at every time of measuring x_i. The inconvenience in this case is that there are five control coefficients to be determined, which will result in some difficulty when some parameters are time dependent intrinsically. In the second and third cases, there are only four coefficients to be determined, but the control ability is not as good as in the first case because the perturbation is added every two and three times of measuring x_i and thus the perturbed system is more sensitive to noise. Although we chose $n = 2$ in the three cases (which is just the same as the dimension of the underlying system), we point out that control is still achievable in the case $n > 2$ though one needs to determine more control coefficients.

Now let us discuss the determination of the control coefficients of the control function in the following three cases.

(a) If the system parameters are time independent, the matrix (16) can be obtained experimentally by the well-known embedding technique and the control coefficients can be obtained as discussed in Sec. II. In this case, one can track the fixed point along an externally adjustable parameter by use of the above perturbation forms.

(b) If the original system is time dependent but the time-dependent parameter (say, b) can be frozen under some conditions (for example, a parameter which is dependent on temperature can be "frozen" at constant temperature), the control coefficients can be obtained for the frozen parameter as in the case (a). And, as shown in the last subsection, the control coefficients obtained for the "frozen" parameter can be used to control the time-dependent system in a range around the value of the parameter as long as the change of b is not too fast.

(c) If the time-dependent parameter b cannot be frozen in any way, one cannot obtain the matrix (16) and thus the determination of the control coefficients is somewhat difficult. A direct way is to search for effective control coefficients in the whole coefficient space. Numerical calculations show that the method is applicable but time wasteful if there are many coefficients to be determined. Let us assume the underlying change of b is random: $b_{i+1} = b_i + \beta\text{ran}(i)$ where ran(i) is a random function whose value is limited to $(0,1)$ and β determines the amplification of the random function. Our numerical calculations find that the control coefficients in a region around $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = (-2.1, -0.25, 0.13, 0, 0)$ can be used to control the time-dependent system when choosing (19). Figure 3(a) shows the result using $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = (-2.1, -0.25, 0.13, 0, 0)$ to implement the control when $\beta = 0.05$. Similarly, we find that the control coefficients in a region around $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-15.5, 3, 2, 0)$ can be used to control the time-dependent system when choosing the perturbation (20). Figure 3(b) shows the result using $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-15.5, 3, 2, 0)$ to implement the control when $\beta = 0.001$. In this example,

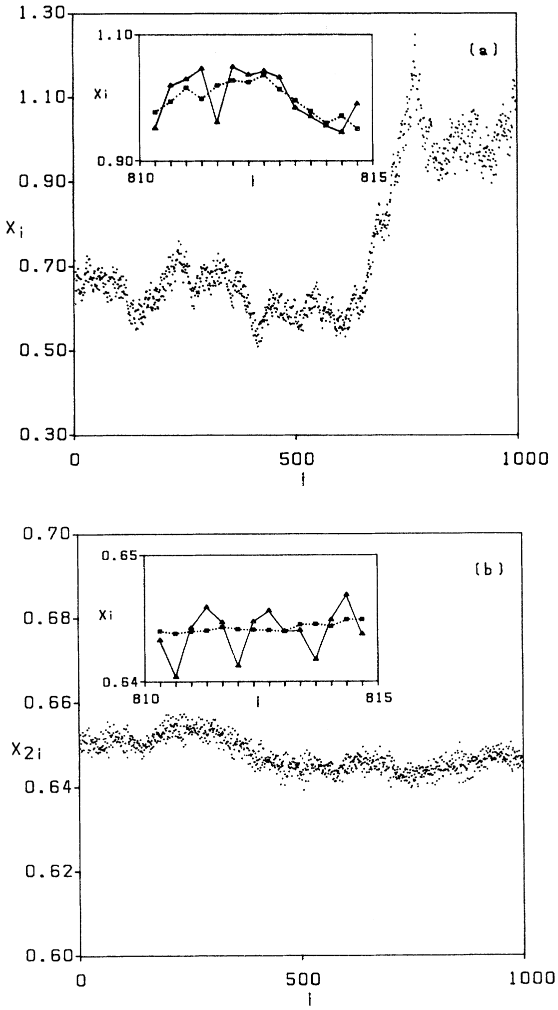

FIG. 3. The numerical results of controlling the time-dependent experimental system when the underlying change of b is $b(i+1) = b(i) + \beta\text{ran}(i)$. The initial value of b is set at 0.3. (a) shows the result of tracking the fixed point (x_+, y_+) by use of the control function (19) with $\beta = 0.05$ and (b) shows the tracking result by use of the control function (20), with $\beta = 0.001$. The trajectory of the perturbed system is denoted by the discrete points (the initial values are set at $x = x_+$ and $y = y_+$ when $b = 0.3$ and $a = 1.29$). The enlarged segments from $i = 810$ to $i = 815$ are set at the upper-left side in which those squares joined by the solid line are the trajectories of the perturbed system. As a contrast, we plot the corresponding location of $x_+(i)$ by those squares linked by the dashed line in the figures.

the last control coefficient of control functions (19) and (20) can always be fixed at 0 because of the special quality of the Hénon map, i.e., the characteristic $y_{i+1} = x_i$.

B. Controlling chaos by making use of delayed information

In the case where prompt feedback is not accessible, one must find ways to implement the control by use of earlier information $\mathbf{z}_{i-l}$ and p_{i-l} as feedback information. To obtain an efficient control function g using the method of this paper, a direct way is to consider the following system:

$$\begin{aligned} \mathbf{z}_{i+1} &= \mathbf{f}^{(l+1)}(\mathbf{z}_{i-l}, p_{i-l}), \\ p_{i+1} &= g(\mathbf{z}_{i-l}, p_{i-l}), \end{aligned} \tag{22}$$

where l denotes the delayed time and $\mathbf{f}^{(l)}$ denotes the lth iterate of $\mathbf{f}$. The function g that makes the fixed point of (22) stable can be determined by the method of Sec. II. Thus one can use the information of time $i - l$ as the feedback information of time $i + 1$ to control the system. However, the method is sensitive to noise when $l \geq 1$ because the perturbation is added to the system at every lth iterate of $\mathbf{f}$.

To illustrate the influence of noise, we add white noise to the Hénon map: $x_{i+1} = 1 - ax_i^2 + by_i + \delta_{xi}$ and $y_{i+1} = x_i + \delta_{yi}$. First, we use prompt information x_{i+1} and y_{i+1} to control the fixed point (x_+, y_+). The control function is

$$a_{i+1} = a_0 + \varepsilon_1(x_{i+1} - x_+) + \varepsilon_2(y_{i+1} - y_+). \tag{23}$$

Figure 4(a) shows the numerical result obtained by using the "center" point of $a = 1.29$ and $b = 0.3$. In this figure, the amplitude of the perturbations is limited to 0.2 and the amplitude of the noise is limited to 3.8×10^{-2}. One can see that control is achieved but there exist some sporadic bursts. In fact, if the amplitude of the noise is less than 3.8×10^{-2}, control is completely achieved. So we think 3.8×10^{-2} is the maximum amplitude of noise that the perturbed system can resist. Second, we use the state of the last time x_i and y_i (i.e., the case $l = 0$) to control the same fixed point. The control function is

$$a_{i+1} = a_0 + \varepsilon_1(x_i - x_+) + \varepsilon_2(y_i - y_+). \tag{24}$$

Figure 4(b) shows that the maximum amplitude of noise that the perturbed system can resist is 1.5×10^{-2}. In the figure, all other data are the same as in Fig. 4(a). It shows that the power to resist noise by use of information of the last time as feedback and the power to resist noise by use of prompt information as feedback are at the same level. However, our numerical calculations show that if one uses earlier ($l \geq 1$) information as feedback, the system is much more sensitive to noise. For example, if $l = 1$, the maximum amplitude of noise that the perturbed system can resist is less than 10^{-3}. The reason is that in this case the perturbations are added every $l + 1$ iterates of $\mathbf{f}$.

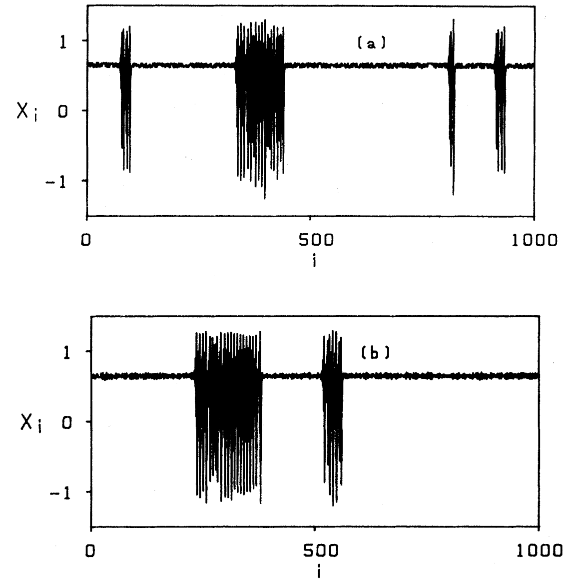

FIG. 4. Controlling chaos by making use of earlier states as feedback information. (a) Using the prompt state x_{i+1} and y_{i+1} as feedback information, the amplitude of the perturbations is limited to 0.2 and the amplitude of noise is limited to 3.8×10^{-2}. (b) Using the earlier state x_i and y_i as feedback information, the maximum amplitude of noise that the perturbed system can resist is 1.5×10^{-2}.

IV. CONCLUSION

The generalized method provides a mathematical framework for describing allowable control functions. It is actually an extension of the "pole placement technique." The extension permits one to choose various control functions instead of only the prompt feedbacks. Among the allowable control functions, those not including the coordinates of the control object provide a natural way to go about tracking, especially when the change of the parameters is involuntary. Another benefit is that the method permits one to control chaos by using earlier states of the system as feedback information in the case where prompt feedback is inaccessible.

The method can be conveniently applied to an experimental system. In the case that the experimental system is time dependent, if the time-dependent parameters can be "frozen" at some conditions, the determination of the control coefficients is the same as in the time-independent case; if "freezing" is impossible, the determination is somewhat difficult. Though a technique to find the control coefficients is suggested in this paper, we still hope more convenient methods will appear.

The control law for higher period orbits, say, a period-N orbit, is principally implementable by considering the

Nth iterate of the map. One may suspect that control functions which do not include the coordinates of the desired object may result in some confusion, i.e., how can the control procedure distinguish different orbits with the same period? In fact, the information of the desired orbit is implicitly included in the control coefficients, so once the control is turned on for the desired orbit in its neighborhood tracking the "correct" orbit should not be an issue. However, the sensitivity to noise when involved at higher periods is also a problem. We will discuss the problem elsewhere.

In addition, we would like to point out that the control functions which do not include the control object in fact provide a way to find unstable periodic orbits of experimental systems.

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[1] E. Ott, C. Grebogi, and A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990); **64**, 2837(E) (1990).
[2] F.J. Romeiras, C. Grebogi, E. Ott, and W.P. Dayawansa, Physica D **58**, 165 (1992).
[3] T. Carroll, I. Triandaf, I. Schwartz, and L. Pecora, Phys. Rev. A **46**, 6189 (1992); Z. Gills, C. Iwata, R. Roy, I. Schwartz, and I. Triandaf, Phys. Rev. Lett. **69**, 3169 (1992).
[4] F. Takens, *Dynamical Systems and Turbulence* (Springer-Verlag, Berlin, 1981), p. 230.
[5] N.H. Packard, J.P. Crutchfield, J.D. Farmer, and R.S. Shaw, Phys. Rev. Lett. **45**, 712 (1980).
[6] U. Dressler and G. Nitsche, Phys. Rev. Lett. **68**, 1 (1992).
[7] M. Hénon, Commun. Math. Phys. **50**, 69 (1976).